Control and Estimation in the Presence of Sporadic Measurements: A Hybrid Systems Approach

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Outline

1. Context
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2. Recent Contributions

3. Robust State Estimation
   ▶ Modeling
   ▶ Hybrid Model

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   ▶ Construction of the functions $V_1$ and $V_2$
   ▶ Observer Design
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5. Numerical examples
Control and estimation under limited information constraints

Main objective

Design control laws and estimation strategies while directly accounting the limitations in the information/resources available

- What kind of information/communication constraints?
  - Quantization
  - Sporadic availability of resources
  - Sampling

- Limited information constraints can affect:
  - The input channel
  - The output channel

To achieve this objective one needs to:

- Include the presence of communication constraints throughout the modeling stage
Context

Control and estimation under limited information constraints

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  - Sporadic availability of resources
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Driving Question

How do we design control and estimation algorithms when information is available sporadically?
Many of the most interesting recent applications pertain to systems linked together through data networks. Some examples:

- UAVs
- Smart grids
- Internet of Things...

Network constraints can impact performance: instability, poor convergence properties...
Inter-vehicles communication affected by packet drops and delays.

- Need of robustness guarantees to these phenomena to avoid collisions in real scenarios
Acknowledgments

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Toulouse, France

Frédéric Gouaisbaut  
LAAS-CNRS,  
Toulouse, France
Different Scenarios

Plant

Controller

\[ y(t_k) \]

\[ y_p \]

\[ y \]

\[ \eta \]
Different Scenarios

\[ y(t_k) \]

- Intersample Correction
- Plant
- Controller

\[ \eta \]

\[ y_p + y \]

\[ t_k \]

\[ y(t_k) \]
Different Scenarios

Intermittency leads to the interplay of continuous-time dynamics and instantaneous changes.
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Closed-loop system modeled as a Hybrid System.
Research focus mostly on Observer-based Control architectures
State estimation in the presence of sporadic measurements

\[ \dot{z} = A z \]
\[ y(t_k) = C z(t_k) + \eta \]

\( y \) is measured at isolated **aperiodic** time instances

\[ t_{k+1} - t_k \in [T_1, T_2] \]
Recent Contributions

State estimation in the presence of sporadic measurements

[F. F., Gouaisbaut, Sanfelice, Tarbouriech, IFAC WC 14, AUT 16]

\[
\dot{z} = Az
\]

\[
y(t_k) = Cz(t_k) + \eta
\]

\(y\) is measured at isolated aperiodic time instances

\[
t_{k+1} - t_k \in [T_1, T_2]
\]

- Continuous-discrete observer [Dinh 2015, Andrieu 2010]

\[
\dot{\hat{z}} = A\hat{z}
\]

\[
\hat{z}(t_{k}^+) = \hat{z}(t_{k}) + L(y(t_k) - C\hat{z}(t_k))
\]
Recent Contributions

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- Continuous-discrete observer [Dinh 2015, Andrieu 2010]

\[ \dot{\hat{z}} = A\hat{z} \]
\[ \hat{z}(t_k^+) = \hat{z}(t_k) + L(y(t_k) - C\hat{z}(t_k)) \]

- Error dynamics modeled as a Hybrid System in the framework of [Goebel et al., PUP 2012]

- State estimation recast as a set stabilization problem

- Lyapunov-based design of the observer gain
Recent Contributions

State estimation in the presence of sporadic measurements

[F. F., Gouaisbaut, Sanfelice, Tarbouriech, IFAC WC 14, AUT 16]

\[ \dot{z} = Az \]

\[ A = \begin{bmatrix} 0.1 & 1 \\ -1 & 0.1 \end{bmatrix}, \quad C = [1 \ 0], \quad L = \begin{bmatrix} 1.0004 \\ -0.0645 \end{bmatrix}, \quad T_1 = 0.8, \ T_2 = 2.3 \]
Recent Contributions

Obs.-based control with intermittent measurements and input access [F. F., Gouaisbaut, Sanfelice, Tarbouriech, ACC 15]

- $y$ is measured at isolated \textit{aperiodic} time instances $t_k$
  
  $$t_{k+1} - t_k \in [T_1^O, T_2^O]$$

- the input channel can be accessed at isolated \textit{aperiodic} time instances $s_k$
  
  $$s_{k+1} - s_k \in [T_1^K, T_2^K]$$
Recent Contributions

Obs.-based control with intermittent measurements and input access [F. F., Gouaisbaut, Sanfelice, Tarbouriech, ACC 15]

- $y$ is measured at isolated aperiodic time instances $t_k$
  \[ t_{k+1} - t_k \in [T_1^O, T_2^O] \]

- the input channel can be accessed at isolated aperiodic time instances $s_k$
  \[ s_{k+1} - s_k \in [T_1^K, T_2^K] \]

- Continuous-discrete observer + state feedback control
- Closed-loop sys. modeled as a hybrid sys. [Goebel et al., PUP 2012]

- Establishment of a Separation Principle
Small delays can destroy the convergence properties of the estimation algorithm.

The evolution of the estimation error with a delay $T^d = 0.002$ (left). The estimation error plot above the time delay lengths for measurements (right). In this case $A = M = 1, T_1 = T_2 = 1$. 
Taking into account delays

- Small delays can destroy the convergence properties of the estimation algorithm

The evolution of the estimation error with a delay $T^d = 0.002$ (left). The estimation error plot above the time delay lengths for measurements (right). In this case $A = M = 1$, $T_1 = T_2 = 1$.

By assuming time-stamped measurements, delays can be fully compensated by the observer! [Guarro, F. F., Sanfelice, NECSYS 18].
Taking into account delays

- Small delays can destroy the convergence properties of the estimation algorithm

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The evolution of the estimation error with a delay $T_d = 0.002$ (left). The estimation error plot above the time delay lengths for measurements (right). In this case $A = M = 1, T_1 = T_2 = 1$.

By assuming time-stamped measurements, delays can be fully compensated by the observer!

[Guarro, F. F., Sanfelice, NECSYS 18]

Need of synchronous clocks on both ends.
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<table>
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<th>General Problem</th>
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<td>Generate <em>estimates</em> of the state variables from noisy sporadic measurements</td>
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**General Problem**

Generate *estimates* of the state variables from noisy sporadic measurements

**Desired Features**

- *Fast rate of convergence*  
  \( (GES, \text{ etc.}) \)
- *Robustness to noise and perturbations*  
  \( (ISS, \text{ rejection, etc.}) \)
- *Quantifiable performance*  
  \( (\text{convergence rate, } \mathcal{L}_p \text{ gain, etc.}) \)
Consider the following family of continuous-time nonlinear plants:

\[
\begin{align*}
\dot{z} &= Az + B\psi(Sz) + Nw \\
y &= Cz + \eta
\end{align*}
\]

\(z \in \mathbb{R}^n\) is the state, \(y \in \mathbb{R}^q\) is the measured output, \(\psi\) is a Lipschitz function with Lipschitz constant \(\ell > 0\), i.e., for all \(v_1, v_2 \in \text{dom } \psi\):

\[
|\psi(v_1) - \psi(v_2)| \leq \ell |v_1 - v_2|
\]

\(w \in \mathbb{R}^p\) and \(\eta \in \mathbb{R}^q\) are, respectively, a nonmeasurable exogenous input and the measurement noise. Matrices \(A, B, S, N,\) and \(C\) are given and with suitable dimensions. Assume:
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- \(y\) is available only at some times \(t_k, k \in \mathbb{N}\), not known.
Consider the following family of continuous-time nonlinear plants

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- \(y\) is available only at some times \(t_k, k \in \mathbb{N}\), not known
- The time elapsed in between measurements is not constant, i.e., any periodicity assumption
State Estimation with Sporadic Measurements

Consider the following family of continuous-time nonlinear plants

\[
\dot{z} = A z + B \psi(S z) + N w
\]
\[
y(t_k) = C z(t_k) + \eta
\]

\(z \in \mathbb{R}^n\) is the state, \(y \in \mathbb{R}^q\) is the measured output, \(\psi\) is a Lipschitz function with Lipschitz constant \(\ell > 0\), i.e., for all \(v_1, v_2 \in \text{dom} \psi\)

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- The time elapsed in between measurements is not constant, i.e., any periodicity assumption
- Just a few mild assumptions:
Consider the following family of continuous-time nonlinear plants

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\[ |\psi(v_1) - \psi(v_2)| \leq \ell |v_1 - v_2| \]

\( w \in \mathbb{R}^p \) and \( \eta \in \mathbb{R}^q \) are, respectively, a nonmeasurable exogenous input and the measurement noise. Matrices \( A, B, S, N, \) and \( C \) are given and with suitable dimensions. Assume:

◮ \( y \) is available only at some times \( t_k \), \( k \in \mathbb{N} \), not known
◮ The time elapsed in between measurements is not constant, i.e., any periodicity assumption
◮ Just a few mild assumptions:
   ◦ \( \{t_k\}_{k=1}^{\infty} \) is strictly increasing and unbounded and \( \exists T_1, T_2 \in \mathbb{R}_{>0} \)
   \[ 0 \leq t_1 \leq T_2 \]
   \[ T_1 \leq t_{k+1} - t_k \leq T_2 \quad \forall k \in \mathbb{N}. \]
   ◦ The arrival of incoming measurements can be detected
State Estimation with Sporadic Measurements

Consider the following family of continuous-time nonlinear plants

\[
\dot{z} = Az + B\psi(Sz) + Nw \\
y(t_k) = Cz(t_k) + \eta
\]

Problem (Robust Estimation)

Design an observer providing an estimate \( \hat{z} \) of \( z \), such that the following three properties are fulfilled:

\((P_1)\) The estimation error converges to zero exponentially with a prescribed convergence rate whenever the input \( w \) and \( \eta \) are identically zero;

\((P_2)\) The estimation error is bounded when the disturbances \( w \) and \( \eta \) are bounded;

\((P_3)\) \( \mathcal{L}_2 \)-external stability from the input \( w \) to a given performance output \( y_p \) is ensured with a prescribed \( \mathcal{L}_2 \)-gain when \( \eta \equiv 0 \).
Related Work

Periodic sampling case

- Observer-protocol pair for LTI networked systems
  [Dacic and Nesic AUT 08]
- Discrete observer for LTI networked systems
  [Park and Martins CDC 12]
- Continuous-time observer for a class of nonlinear systems
  [Ahmed-Ali ea. SCL 13]

Continuous-discrete observer

- Lipschitz continuous-time systems of small dimensions using reachable sets (periodic)
  [Farza ea. TAC 14] [Dinh ea. TAC 15]
- Impulsive systems approach (periodic)
  [Mazenc ea. SIAM 15]
- Hybrid systems approach (linear case)
  [Ferrante ea. AUT 16]
- Hybrid systems approach (nonlinear case)
  [Etienne ea. AUT 17]
Observer Algorithm

\[
\begin{align*}
\dot{z}(t) &= A\dot{z}(t) + B\psi(S\dot{z}(t)) + L\theta(t) \\
\dot{\theta}(t) &= H\theta(t) \\
\Rightarrow \\
\hat{z}(t^+) &= \hat{z}(t) \\
\hat{\theta}(t^+) &= y(t) - C\hat{z}(t)
\end{align*}
\]

\forall t \neq t_k, k \in \mathbb{N}_{>0}

\forall t = t_k, k \in \mathbb{N}_{>0}

\Rightarrow \text{the gains } L \text{ and } H \text{ are to be designed.}
Observer Algorithm

The operating principle of the observer is as follows:

- The arrival of a new measurement triggers an instantaneous jump in the observer state and the measured output estimation error, i.e., $e_y := y - C\hat{z}$, is instantaneously stored in $\theta$;

- in between consecutive measurements, $\theta$ is continuously updated according to continuous-time dynamics, and its value is continuously used as an intersample correction to feed a continuous-time observer.

\[
\begin{align*}
\dot{\hat{z}}(t) &= A\hat{z}(t) + B\psi(S\hat{z}(t)) + L\theta(t) \\
\dot{\theta}(t) &= H\theta(t) \quad \forall t \neq t_k, k \in \mathbb{N}_{>0} \\
\hat{z}(t^+) &= \hat{z}(t) \\
\theta(t^+) &= y(t) - C\hat{z}(t) \quad \forall t = t_k, k \in \mathbb{N}_{>0}
\end{align*}
\]

- the gains $L$ and $H$ are to be designed.
Observer Algorithm

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\]  \quad \forall t = t_k, k \in \mathbb{N}_{>0}

- the gains $L$ and $H$ are to be designed.

A quick literature review...

- $H = 0 \implies$ Sample-and-hold [Raff ea. ACC 2007]
- $H = -CL \implies$ predictor-based [Karafyllis ea. TAC 2009]

Picture from [Karafyllis ea. TAC 2009]
The operating principle of the observer is as follows:

- The arrival of a new measurement triggers an instantaneous jump in the observer state and the measured output.

- \( H = 0 \) \( \implies \) Sample-and-hold [Raff ea. ACC 2007]

- \( H = -CL \) \( \implies \) predictor-based [Karafyllis ea. TAC 2009]

Existing approaches rely on certain structures of \( H \) to derive design results for \( L \).

We design \( L \) and \( H \) in one shot!
Consider the following (invertible) change of variables:

\[ \varepsilon := z - \hat{z} \]  

estimation error

\[ \tilde{\theta} := C'(z - \hat{z}) - \theta \]  

estimation error of the output estimation error

and let us take as a performance output \( y_p = C_p \varepsilon \), for some \( C_p \)

\[
\begin{aligned}
    \dot{z}(t) &= Az(t) + G\psi(Sz(t)) + Nw(t) \\
    \begin{pmatrix}
        \dot{\varepsilon}(t) \\
        \dot{\tilde{\theta}}(t)
    \end{pmatrix} &= \mathcal{F}
    \begin{pmatrix}
        \varepsilon(t) \\
        \tilde{\theta}(t)
    \end{pmatrix}
    + Q\zeta(z(t), \varepsilon(t)) + T w(t) & \forall t \neq t_k \\
    \begin{pmatrix}
        z(t^+) \\
        \varepsilon(t^+) \\
        \tilde{\theta}(t^+)
    \end{pmatrix} &=
    \begin{pmatrix}
        z(t) \\
        \varepsilon(t) \\
        \tilde{\theta}(t)
    \end{pmatrix} + \mathcal{N} \eta(t) & \forall t = t_k \\
    y_p(t) &= C_p \varepsilon(t)
\end{aligned}
\]

where for each \( v_1, v_2 \in \mathbb{R}^{nz} \), \( \zeta(v_1, v_2) := \psi(Sv_1) - \psi(S(v_1 - v_2)) \)

and

\[
\begin{align*}
    \mathcal{F} &:= \begin{pmatrix}
        A - LC \\
        CA - CLC - HC \\
        CA - CLC - HC
    \end{pmatrix}
    \begin{pmatrix}
        L \\
        CL + H \\
        CL + H
    \end{pmatrix},
    \mathcal{T} &:= \begin{pmatrix}
        N \\
        CN
    \end{pmatrix},
    \mathcal{Q} &:= \begin{pmatrix}
        B \\
        CB
    \end{pmatrix},
    \mathcal{G} &:= \begin{pmatrix}
        I & 0 \\
        0 & 0
    \end{pmatrix},
    \mathcal{N} &:= \begin{pmatrix}
        0 \\
        -I
    \end{pmatrix}
\end{align*}
\]
Consider the following (invertible) change of variables:

\[
\begin{align*}
\varepsilon &:= z - \hat{z} \quad \text{estimation error} \\
\tilde{\theta} &:= C(z - \hat{z}) - \theta \quad \text{estimation error of the output estimation error}
\end{align*}
\]

and let us take as a performance output \( y_p = C_p \varepsilon \), for some \( C_p \)

\[
\begin{align*}
\dot{z}(t) &= Az(t) + G\psi(Sz(t)) + Nw(t) \\
\begin{bmatrix} \dot{\varepsilon}(t) \\ \dot{\theta}(t) \end{bmatrix} &= \mathcal{F} \begin{bmatrix} \varepsilon(t) \\ \theta(t) \end{bmatrix} + \mathcal{Q}\zeta(z(t), \varepsilon(t)) + \mathcal{T}w(t) & \forall t \neq t_k \\
\begin{bmatrix} z(t^+) \\ \varepsilon(t^+) \\ \theta(t^+) \end{bmatrix} &= \begin{bmatrix} z(t) \\ \varepsilon(t) \\ \theta(t) \end{bmatrix} + \mathcal{G}\zeta(z(t), \varepsilon(t)) + \mathcal{N}\eta(t) & \forall t = t_k \\
y_p(t) &= C_p\varepsilon(t)
\end{align*}
\]

where for each \( v_1, v_2 \in \mathbb{R}^{n_z} \), \( \zeta(v_1, v_2) := \psi(Sv_1) - \psi(S(v_1 - v_2)) \)

- The error dynamics manifest both continuous time behavior and instantaneous jumps
- We shall model such dynamics as a hybrid system
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Modeling Hybrid Systems: **Flow + Jump**

\[ \begin{align*}
H \quad \left\{ \begin{array}{c}
\dot{x} & \in F(x, u) \\
x^+ & \in G(x, u) \\
y & = h(x)
\end{array} \right. \\
\text{where } x \text{ is the state, } u \text{ the input, } y \text{ the output}
\end{align*} \]

- \( C \) is the **flow set**
- \( F \) is the **flow map**
- \( D \) is the **jump set**
- \( G \) is the **jump map**

and \( h \) is the **output map.**
Modeling Hybrid Systems: **Flow + Jump**

\[
\begin{align*}
\mathcal{H} \quad \begin{cases} 
\dot{x} & \in F(x, u) \quad x \in \mathcal{C} \subset \mathbb{R}^n \\
 x^{+} & \in G(x, u) \quad x \in \mathcal{D} \subset \mathbb{R}^n \\
y & = h(x)
\end{cases}
\end{align*}
\]

where \( x \) is the state, \( u \) the input, \( y \) the output

- \( \mathcal{C} \) is the *flow set*
- \( \mathcal{D} \) is the *jump set*
- \( F \) is the *flow map*
- \( G \) is the *jump map*

and \( h \) is the *output map*.

Solutions are functions parameterized by hybrid time \((t, j)\):

- Flows parameterized by \( t \in \mathbb{R}_{\geq 0} := [0, +\infty) \)
- Jumps parameterized by \( j \in \mathbb{N}_{\geq 0} := \{0, 1, 2, \ldots\} \)
Modeling Hybrid Systems: Flow + Jump

\[
\mathcal{H} \begin{cases} 
  \dot{x} & \in F(x, u) \quad x \in \mathcal{C} \subset \mathbb{R}^n \\
  x^+ & \in G(x, u) \quad x \in \mathcal{D} \subset \mathbb{R}^n \\
  y & = h(x)
\end{cases}
\]

where \( x \) is the state, \( u \) the input, \( y \) the output

- \( \mathcal{C} \) is the flow set
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- Flows parameterized by \( t \in \mathbb{R}_{\geq 0} := [0, +\infty) \)
- Jumps parameterized by \( j \in \mathbb{N}_{\geq 0} := \{0, 1, 2, \ldots\} \)

Then, solutions to \( \mathcal{H} \) are given by pairs \((x, u)\) defined on (hyb. time domain)

\([0, t_1] \times \{0\}\)
Modeling Hybrid Systems: Flow + Jump

\[ \mathcal{H} \begin{cases} \dot{x} & \in F(x, u) \quad x \in C \subset \mathbb{R}^n \\ x^+ & \in G(x, u) \quad x \in D \subset \mathbb{R}^n \\ y & = h(x) \end{cases} \]

where \( x \) is the state, \( u \) the input, \( y \) the output

- \( C \) is the flow set
- \( D \) is the jump set
- \( F \) is the flow map
- \( G \) is the jump map

and \( h \) is the output map.

Solutions are functions parameterized by hybrid time \( (t, j) \):

- **Flows** parameterized by \( t \in \mathbb{R}_{\geq 0} := [0, +\infty) \)
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Then, solutions to \( \mathcal{H} \) are given by pairs \( (x, u) \) defined on (hyb. time domain)

\[ ([0, t_1] \times \{0\}) \cup ([t_1, t_2] \times \{1\}) \]
Modeling Hybrid Systems: \textbf{Flow + Jump}

\[ \mathcal{H} \left\{ \begin{array}{l}
\dot{x} \in F(x,u) \\
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y = h(x)
\end{array} \right. \]

where \( x \) is the \textit{state}, \( u \) the \textit{input}, \( y \) the \textit{output}

- \( \mathcal{C} \) is the \textit{flow set}
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- \( F \) is the \textit{flow map}
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Solutions are functions parameterized by hybrid time \((t, j)\):

- \textbf{Flows} parameterized by \( t \in \mathbb{R}_{\geq 0} := [0, +\infty) \)
- \textbf{Jumps} parameterized by \( j \in \mathbb{N}_{\geq 0} := \{0, 1, 2, \ldots\} \)

Then, solutions to \( \mathcal{H} \) are given by pairs \((x, u)\) defined on (hyb. time domain)

\[ ([0, t_1] \times \{0\}) \cup ([t_1, t_2] \times \{1\}) \cup \ldots ([t_j, t_{j+1}] \times \{j\}) \cup \ldots \]
Modeling Hybrid Systems: \textbf{Flow} \ + \ \textbf{Jump}

\[
\mathcal{H} \left\{ \begin{array}{l}
\dot{x} \in F(x, u) \quad x \in \mathcal{C} \subset \mathbb{R}^n \\
x^+ \in G(x, u) \quad x \in \mathcal{D} \subset \mathbb{R}^n \\
y = h(x)
\end{array} \right.
\]

where \( x \) is the state, \( u \) the input, \( y \) the output

\begin{itemize}
  \item \( \mathcal{C} \) is the flow set
  \item \( F \) is the flow map
  \item \( \mathcal{D} \) is the jump set
  \item \( G \) is the jump map
\end{itemize}

and \( h \) is the output map.

Solutions are functions parameterized by hybrid time \((t, j)\):

\begin{itemize}
  \item Flows parameterized by \( t \in \mathbb{R}_{\geq 0} := [0, +\infty) \)
  \item Jumps parameterized by \( j \in \mathbb{N}_{\geq 0} := \{0, 1, 2, \ldots\} \)
\end{itemize}

Then, solutions to \( \mathcal{H} \) are given by pairs \((x, u)\) defined on (hyb. time domain)

\[
([0, t_1] \times \{0\}) \cup ([t_1, t_2] \times \{1\}) \cup \ldots ([t_j, t_{j+1}] \times \{j\}) \cup \ldots
\]

The state \( x \) can have logic, memory, and timer components.
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Time-varying and (potentially) stochastic system can be modeled as an autonomous hybrid inclusion
Time-varying and (potentially) stochastic system can be modeled as an autonomous hybrid inclusion.

Idea: To capture all possible event sequences \( \{t_k\}_{k=1}^{\infty} \) while removing dependency on time and stochastic dynamics, define

\( \tau \) as a timer that, when expires, generates the sampling events.
Estimation with Sporadic Measurements—Modeling

Time-varying and (potentially) stochastic system can be modeled as an autonomous hybrid inclusion

Idea: To capture all possible event sequences \( \{t_k\}_{k=1}^{\infty} \) while removing dependency on time and stochastic dynamics, define

- \( \tau \) as a timer that, when expires, generates the sampling events

\[
\begin{align*}
\dot{\tau} &= -1 & \tau &\in [0, T_2] \\
\tau^+ &\in [T_1, T_2] & \tau &= 0
\end{align*}
\]
Time-varying and (potentially) stochastic system can be modeled as an autonomous hybrid inclusion.

Idea: To capture all possible event sequences $\{t_k\}_{k=1}^\infty$ while removing dependency on time and stochastic dynamics, define

$\tau$ as a timer that, when expires, generates the sampling events

$$
\begin{cases}
\dot{\tau} = -1 & \tau \in [0, T_2] \\
\tau^+ \in [T_1, T_2] & \tau = 0
\end{cases}
$$
Hybrid modeling

Hybrid system $\mathcal{H}_e$ with: state $x := (z, \varepsilon, \tilde{\theta}, \tau) \in \mathbb{R}^{2n+q+1}$, input $u = (w, \eta) \in \mathbb{R}^{p+q}$, and output $y_p$:

$$
\mathcal{H}_e \left\{ \begin{array}{l}
\dot{x} = f(x, w) \quad x \in \mathcal{C}, w \in \mathbb{R}^{p} \\
x^+ \in G(x, \eta) \quad x \in \mathcal{D}, \eta \in \mathbb{R}^{q} \\
y_p = C_p \varepsilon 
\end{array} \right.
$$

where

$$f(x, w) = \begin{pmatrix}
Az + B\psi(Sz) + Nw \\
\mathcal{F} \left( \varepsilon, \tilde{\theta} \right) + Q\zeta(z, \varepsilon) + T_w \\
\mathcal{G} \left( \varepsilon, \tilde{\theta} \right) + N\eta
\end{pmatrix} \quad \forall x \in \mathcal{C}, w \in \mathbb{R}^{p}$$

$$G(x, \eta) = \begin{pmatrix}
z \\
\mathcal{G} \left( \varepsilon, \tilde{\theta} \right) + N\eta
\end{pmatrix} \quad \forall x \in \mathcal{D}, \eta \in \mathbb{R}^{q}$$

and the flow set $\mathcal{C}$ and the jump set $\mathcal{D}$ are defined as follows

$$\mathcal{C} = \mathbb{R}^{2n+q} \times [0, T_2], \quad \mathcal{D} = \mathbb{R}^{2n+q} \times \{0\}$$
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**Sufficient conditions**

**Assumption (□)**

Let $\lambda_t$ and $\gamma$ be given positive real numbers. There exist two continuously differentiable functions $V_1 : \mathbb{R}^{n_z} \to \mathbb{R}$, $V_2 : \mathbb{R}^{n_y+1} \to \mathbb{R}$, positive real numbers $\alpha_1, \alpha_2, \omega_1, \omega_2$ such that

(A1) $\alpha_1 |\varepsilon|^2 \leq V_1(\varepsilon) \leq \alpha_2 |\varepsilon|^2 \quad \forall x \in \mathcal{C}$;

(A2) $\omega_1 |\tilde{\theta}|^2 \leq V_2(\tilde{\theta}, \tau) \leq \omega_2 |\tilde{\theta}|^2 \quad \forall x \in \mathcal{C}$;

(A3) the function $x \mapsto V(x) := V_1(\varepsilon) + V_2(\tilde{\theta}, \tau)$ satisfies for each $x \in \mathcal{C}, w \in \mathbb{R}^{n_w}$

$$
\left\langle \nabla V(x), \begin{pmatrix}
Az + B\psi(Sz) + Nw \\
\mathcal{F} \left( \begin{pmatrix} \varepsilon \\ \tilde{\theta} \end{pmatrix} \right) + Q\zeta(z, \varepsilon) + Tw \\
-1
\end{pmatrix} \right\rangle \leq -2\lambda_t V(x) - \varepsilon^T C_p^T C_p \varepsilon + \gamma^2 w^T w
$$
Let us define

$$\mathcal{A} := \mathbb{R}^n \times \{0\} \times \{0\} \times [0, T_2]$$

- The (closed) set $\mathcal{A}$ contains the set of points in which $\varepsilon = 0$ and $\tilde{\theta} = 0$
- We let the timer variable evolve in $[0, T_2]$
Let us define
\[ \mathcal{A} := \mathbb{R}^n \times \{0\} \times \{0\} \times [0, T_2] \]

The (closed) set \( \mathcal{A} \) contains the set of points in which \( \varepsilon = 0 \) and \( \tilde{\theta} = 0 \).

**Theorem**

Let Assumption \( \Box \) hold. Then\(^1\):

(i) There exists \( \vartheta \in \mathbb{R}_{>0} \) such that for each maximal solution to \( \mathcal{H}_e \) of the form \((\phi, 0)\), one has
\[ |\phi(t, j)|_\mathcal{A} \leq \vartheta e^{-\lambda t} |\phi(0, 0)|_\mathcal{A} \quad \forall (t, j) \in \text{dom} \phi \]

(ii) The hybrid system \( \mathcal{H}_e \) is **input-to-state stable** with respect to \( \mathcal{A} \) relatively to \( u = (w, \eta) \);

(iii) There exists \( \alpha > 0 \) such that any solution pair \((\phi, u)\) to \( \mathcal{H}_e \) with \( \eta \equiv 0 \) satisfies
\[ \sqrt{\int_{\mathcal{I}} |y_p(s, j(s))|^2 \, ds} \leq \alpha |\phi(0, 0)|_\mathcal{A} + \gamma \sqrt{\int_{\mathcal{I}} |w(s, j(s))|^2 \, ds} \]
Sketch of the proof.

Consider the following Lyapunov function candidate \cite{Goebel ea. 2007 CSM}

\[ \mathbb{R}^{2n+q} \times \mathbb{R}_{\geq 0} \ni x \mapsto V(x) := V_1(\varepsilon) + V_2(\tilde{\theta}, \tau) \]

Set \( \rho_1 = \min\{\alpha_1, \omega_1\} \) and \( \rho_2 = \max\{\alpha_2, \omega_2\} \). Then, in view of the definition of the set \( \mathcal{A} \)

\[ \rho_1|x|_A^2 \leq V(x) \leq \rho_2|x|_A^2 \quad \forall x \in \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D}) \]

Moreover, from Assumption \( \square \) item (A3), one has

\[ \langle \nabla V(x), f(x, w) \rangle \leq -2\lambda_t V(x) + \gamma^2 w^T w \quad \forall x \in \mathcal{C}, w \in \mathbb{R}^p \quad (\star) \]

and for each \( g \in G(x, \eta), x \in \mathcal{D}, \eta \in \mathbb{R}_q \) one has

\[ V(g) - V(x) = -V_2(\tilde{\theta}, 0) + V_2(-\eta, v) \leq \omega_2|\eta|^2 \quad (\triangle) \]
Cont’d.

\((\star) + (\Delta) \implies \text{for each } (t, j) \in \text{dom} \phi \text{ one has}

\[ |\phi(t, j)|_A \leq \max \left\{ 2\sqrt{\frac{\rho_2}{\rho_1}} e^{-\lambda t} |\phi(0, 0)|_A, 2 \max \left\{ \frac{\gamma}{\sqrt{2\lambda_t \rho_1}}, \sqrt{\omega_2 \frac{e^{4\lambda t T_1}}{e^{2\lambda t T_1 - 1}}} \right\} \|u\|_\infty \right\} \quad (\diamond)

\]

Pick \( u \equiv 0 \) gives:

\[ |\phi(t, j)|_A \leq \sqrt{\frac{\rho_2}{\rho_1}} e^{-\lambda t} |\phi(0, 0)|_A \quad \forall (t, j) \in \text{dom} \phi

\]

that is \((i)\).
(ii) can be easily shown from (◊), by relying on the following lemma

**Lemma**

Let \(\lambda_t > 0, T_1 > 0, \lambda \in \left(0, \frac{\lambda_t T_1}{1+T_1}\right]\), and \(\omega \geq \lambda\). Then, each solution pair \((\phi, u)\) to \(H_e\) satisfies

\[-\lambda_t t \leq \omega - \lambda(t + j)\]

for every \((t, j) \in \text{dom } \phi\).
(ii) can be easily shown from (◇), by relying on the following lemma

**Lemma**

Let $\lambda_t > 0$, $T_1 > 0$, $\lambda \in \left(0, \frac{\lambda T_1}{1+T_1}\right]$, and $\omega \geq \lambda$. Then, each solution pair $(\phi, u)$ to $\mathcal{H}_e$ satisfies

$$-\lambda_t t \leq \omega - \lambda(t + j)$$

for every $(t, j) \in \text{dom} \phi$.

To establish (iii), pick $u = (w, 0)$ and let $(\phi, u)$ be a maximal solution pair to $\mathcal{H}_e$. Pick any $T > 0$, then thanks to Assumption □ item (A3), since $V$ is nonincreasing at jumps, one gets

$$\int_{\mathcal{I}(t)} \varepsilon(s, j(s))^T C_p^T C_p \varepsilon(s, j(s)) ds \leq V(\phi(0, 0))$$

$$+ \gamma^2 \int_{\mathcal{I}(t)} |w(s, j(s))|^2 ds$$

where $\mathcal{I}(t) := [0, t] \cap \text{dom}_t \phi$. By taking the limit for $t$ approaching $\sup_t \text{dom} \phi$, one gets (iii) with $\alpha = \rho_2$. □
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Construction of the functions $V_1$ and $V_2$

**Theorem**

Let $\lambda_t, \gamma \in \mathbb{R}_{>0}$. If there exist $P_1 \in S^n_+, P_2 \in S^q_+, \delta, \chi \in \mathbb{R}_{>0}$, and two matrices $L \in \mathbb{R}^{n \times q}, H \in \mathbb{R}^{q \times q}$, such that

$$M(0) \leq 0, \quad M(T_2) \leq 0 \quad (\heartsuit)$$

where for all $\tau \in [0, T_2]$

$$M(\tau) = \begin{pmatrix}
\text{He}(P_1(A - LC)) + 2\lambda_t P_1 + C_p^T C_p + \chi \ell^2 S^T S & P_1 L + e^{\delta \tau} (CA - CLC - HC)^T P_2 & P_1 N & P_1 B \\
\cdot & e^{\delta \tau} (\text{He}(P_2(CL + H)) + (2\lambda_t - \delta) P_2) & e^{\delta \tau} P_2 CN & e^{\delta \tau} P_2 CB \\
\cdot & \cdot & -\gamma^2 I_{n_w} & 0 \\
\cdot & \cdot & \cdot & -\chi I_{n_s}
\end{pmatrix}$$

Then, the functions $\varepsilon \mapsto V_1(\varepsilon) := \varepsilon^T P_1 \varepsilon$ and $(\tilde{\theta}, \tau) \mapsto V_2(\tilde{\theta}, \tau) := e^{\delta \tau} \tilde{\theta}^T P_2 \tilde{\theta}$ satisfy Assumption $\square$.

*Here $S^q_+$ is the set of positive definite $q \times q$ real matrices.*
Sketch of the proof

Items (A1) and (A2) of Assumption □ are satisfied with

\[ \alpha_1 = \lambda_{\min}(P_1), \omega_1 = \lambda_{\min}(P_2) \]
\[ \alpha_2 = \lambda_{\max}(P_1), \omega_2 = \lambda_{\max}(P_2)e^{\delta T_2} \]
Sketch of the proof

Items (A1) and (A2) of Assumption □ are satisfied with

\[ \alpha_1 = \lambda_{\min}(P_1), \omega_1 = \lambda_{\min}(P_2) \]
\[ \alpha_2 = \lambda_{\max}(P_1), \omega_2 = \lambda_{\max}(P_2)e^{\delta T_2} \]

Define for each \( x \in C, w \in \mathbb{R}^{n_w} \)

\[ \Omega(x, w) := \langle \nabla(V_1(\varepsilon) + V_2(\tilde{\theta}, \tau)), f(x, w) \rangle + \varepsilon^T C_p^T C_p \varepsilon + 2\lambda_t(V_1(\varepsilon) + V_2(\tilde{\theta}, \tau)) \]

From Lipschitzness of \((z, \varepsilon) \mapsto \zeta(z, \varepsilon), x \in C, w \in \mathbb{R}^{n_w} \)

\[ \Omega(x, w) \leq \Omega(x, w) - \chi(\zeta(z, \varepsilon)^T \zeta(z, \varepsilon) - \ell^2 \varepsilon^T S^T S \varepsilon) = \]

\[ \begin{pmatrix} \varepsilon \\ \tilde{\theta} \\ w \\ \zeta(z, \varepsilon) \end{pmatrix}^T \mathcal{M}(\tau) \begin{pmatrix} \varepsilon \\ \tilde{\theta} \\ w \\ \zeta(z, \varepsilon) \end{pmatrix} \]
Sketch of the proof

Items (A1) and (A2) of Assumption □ are satisfied with

\[ \alpha_1 = \lambda_{\min}(P_1), \omega_1 = \lambda_{\min}(P_2) \]
\[ \alpha_2 = \lambda_{\max}(P_1), \omega_2 = \lambda_{\max}(P_2) e^{\delta T_2} \]

Define for each \( x \in C, w \in \mathbb{R}^{n_w} \)

\[ \Omega(x, w) := \langle \nabla(V_1(\varepsilon) + V_2(\tilde{\theta}, \tau)), f(x, w) \rangle + \varepsilon^T C_p^T C_p \varepsilon + 2\lambda_t (V_1(\varepsilon) + V_2(\tilde{\theta}, \tau)) \]

From Lipschitzness of \( (z, \varepsilon) \mapsto \zeta(z, \varepsilon), x \in C, w \in \mathbb{R}^{n_w} \)

\[ \Omega(x, w) \leq \Omega(x, w) - \chi(\zeta(z, \varepsilon)^T \zeta(z, \varepsilon) - \ell^2 \varepsilon^T S^T S \varepsilon) = \]
\[ \begin{pmatrix} \varepsilon \\ \tilde{\theta} \end{pmatrix}^T \begin{pmatrix} \varepsilon \\ \tilde{\theta} \end{pmatrix} \]

Item (A3) of Assumption □ holds if

\[ \mathcal{M}(\tau) \leq 0 \quad \forall \tau \in [0, T_2] \]
To conclude the proof, it can be shown that $\exists \lambda : [0, T_2] \rightarrow [0, 1]$ such that for each $\tau \in [0, T_2]$,

$$M(\tau) = \lambda(\tau)M(0) + (1 - \lambda(\tau))M(T_2)$$

Therefore, it follows that the satisfaction of (♥) implies $M(\tau) \leq 0$ for each $\tau \in [0, T_2]$, concluding the proof.
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Design Issues

- Designing the observer via the above result requires the solution of two nonlinear matrix inequalities! Unlikely from a numerical standpoint.

- Two kinds of nonlinearity in $\mathcal{M}$
  - $\exp(\delta T_2)$ and $\delta P_2$. Easy to manage in practice via grid search over $\delta$.
  - Bilinear terms in $P_1, P_2, L, H$. Hard to manage via congruence transformation due to CLC ($L$ is jammed) and $L$ multiplying both $P_2$ and $P_1$!

We propose four LMI-based design strategies

- “$H$-Linearization”: Select $H$ to cancel out bothering nonlinearities, $H = X - CL$,

$$
\begin{pmatrix}
\text{He}(P_1(A - LC')) & P_1 L + (CA - CLC - HC)^T P_2 \\
\text{He}(P_2(CL + H)) - \delta P_2 & \downarrow
\end{pmatrix}

\begin{pmatrix}
\text{He}(P_1(A - LC')) & P_1 L + (CA - X)^T P_2 \\
\text{He}(P_2X - \delta P_2) & \downarrow
\end{pmatrix}
$$
Design Issues

- Designing the observer via the above result requires the solution of two nonlinear matrix inequalities! Unlikely from a numerical standpoint.

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  - $\exp(\delta T_2)$ and $\delta P_2$. Easy to manage in practice via grid search over $\delta$.
  - Bilinear terms in $P_1, P_2, L, H$. Hard to manage via congruence transformation due to $CLC$ ($L$ is jammed) and $L$ multiplying both $P_2$ and $P_1$.

\[
\begin{pmatrix}
\text{He}(P_1(A - LC)) & P_1L + (CA - X)^TP_2 \\
\bullet & \text{He}(P_2X) - \delta P_2
\end{pmatrix}
\]

$X$ is a new variable to be designed. Then invertible changes of variables lead to LMIs!
Design Issues

- Designing the observer via the above result requires the solution of two nonlinear matrix inequalities! Unlikely from a numerical standpoint

- Two kinds of nonlinearity in $\mathcal{M}$
  - $\exp(\delta T_2)$ and $\delta P_2$. Easy to manage in practice via grid search over $\delta$.
  - Bilinear terms in $P_1, P_2, L, H$. Hard to manage via congruence transformation due to $CLC$ ($L$ is jammed) and $L$ multiplying both $P_2$ and $P_1$!

Remark

- Setting $X = 0$ yields the predictor-based observer in [Karafyllis ea. TAC 2009]!
- First time LMI-based design for the observer in [Karafyllis ea. TAC 2009]
Slack variables-based design

- Slack variables-based design: Derive equivalent conditions to $\mathcal{M}_1 < 0, \mathcal{M}_2 < 0$ via the projection lemma [Gahinet et al. IJRNC 94]. This avoids the term $CLC$ and products between $P_1, P_2$ and $H, L$.

- Then, structuring the slack variables (conservative!) + invertible changes of variables $\rightarrow$ LMIs.

Following this approach we derive two LMI-based design strategies:

- The first leads back to the ZOH sample-and-hold observer in [Raff et al. ACC 2007], i.e., $H = 0$, and provides a novel LMI-based design for such a scheme.

- The second one does not constraint $H$ and leads to LMI-based design for a novel observer scheme!
Slack variables-based design

- Slack variables-based design: Derive equivalent conditions to $M_1 < 0, M_2 < 0$ via the projection lemma \[ \text{[Gahinet et al. IJRC 94]} \]. This avoids the term $CLC$ and products between $P_1, P_2$ and $H,L$.

- Then, structuring the slack variables (conservative!) + invertible changes of variables $\rightarrow$ LMIs.

Following this approach we derive two LMI-based design strategies:

- The first leads back to the ZOH sample-and-hold observer in \[ \text{[Raff et al. ACC 2007]} \], i.e., $H = 0$, and provides a novel LMI-based design for such a scheme.

- The second one does not constraint $H$ and leads to LMI-based design for a novel observer scheme!

**Proposition 3 (Sample-and-hold Implementation).** Let $\lambda, \gamma$ be given positive real numbers. If there exist $P_1 \in S^{n_x}_{++}, P_2 \in S^{n_y}_{++}$, positive real numbers $\delta, \chi$, a nonsingular matrix $X \in \mathbb{R}^{n_x \times n_x}$, and matrices $X_5, X_6, X_7, X_8 \in \mathbb{R}^{n_y \times n_y}, X_9, Y_5, Y_6, Y_7 \in \mathbb{R}^{n_x \times n_y}, J \in \mathbb{R}^{n_x \times n_x}$ such that

\[
\begin{pmatrix}
\text{He}(Q_1) & Q_2 + \mathcal{P} & Q_3 & Q_4 \\
\dot{\mathcal{N}} + \text{He}(Q_5) & Q_6 & Q_7 & Q_8 \\
0 & -\gamma^2 I & 0 & -\chi I \\
\end{pmatrix} < 0
\]

\[
\begin{pmatrix}
\text{He}(\tilde{Q}_1) & \tilde{Q}_2 + \mathcal{P}_{T_2} & Q_3 & Q_4 \\
\dot{\mathcal{N}}_{T_2} + \text{He}(Q_5) & Q_6 & Q_7 & Q_8 \\
0 & -\gamma^2 I & 0 & -\chi I \\
\end{pmatrix} < 0
\]

where $\mathcal{P}, \mathcal{P}_{T_2}, \mathcal{N}, \mathcal{N}_{T_2}$ are defined in (21) and

\[
Q_1 = \begin{pmatrix}
-X + C^T X_5 & C^T X_6 \\
-X_5 & -X_6
\end{pmatrix}
\]

\[
Q_2 = \begin{pmatrix}
-X + X^T A - JC + C^T X_7 & J + C^T X_8 \\
-X_7 & -X_8
\end{pmatrix}
\]

\[
Q_3 = \begin{pmatrix}
X^T N \\
0
\end{pmatrix}
\]

\[
Q_4 = \begin{pmatrix}
X^T B \\
0
\end{pmatrix}
\]

\[
Q_5 = \begin{pmatrix}
A^T X - C^T J^T & 0 \\
J^T & 0
\end{pmatrix}
\]

\[
Q_6 = \begin{pmatrix}
X^T N \\
0
\end{pmatrix}
\]

\[
Q_7 = \begin{pmatrix}
X^T B \\
0
\end{pmatrix}
\]

\[
\tilde{Q}_1 = \begin{pmatrix}
-X + C^T Y_5 & C^T Y_6 \\
-Y_5 & -Y_6
\end{pmatrix}
\]

\[
\tilde{Q}_2 = \begin{pmatrix}
-X + X^T A - JC + C^T Y_7 & J + C^T Y_8 \\
-Y_7 & -Y_8
\end{pmatrix}
\]

then $L = X^{-T} J$ and $H = 0$ are a solution to Problem 1.
Slack-variables based designs are in general more computationally expensive.

Number of scalar variables associated to the different results vs $n$ for $q = 1$:
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The proposed apparatus allows one to include optimization aspects in the design of the observer.

- For example
Optimization Aspects

The proposed apparatus allows one to include optimization aspects in the design of the observer

• For example

\[ \begin{align*}
\text{minimize} & \quad \gamma^2 \\
\text{s.t.} & \quad P_1, P_2, L, H, \gamma, \delta, \chi \\
& \quad P_1 \in S_{nz}^n, P_2 \in S_{ny}^n, \delta > 0, \chi \geq 0 \\
& \quad M(0) \leq 0, M(T_2) \leq 0
\end{align*} \]
The proposed apparatus allows one to include optimization aspects in the design of the observer.

- For example

\[
\text{minimize } P_1, P_2, L, H, \gamma, \delta, \chi \\
\text{s.t. } P_1 \in S_n^{nz}, P_2 \in S_n^{ny}, \delta > 0, \chi \geq 0 \\
M(0) \leq 0, M(T_2) \leq 0
\]
The proposed apparatus allows one to include optimization aspects in the design of the observer.

- For example

\[ \begin{align*}
\text{minimize} & \quad (-T_2, \gamma^2) \\
\text{s.t.} & \quad P_1, P_2, L, H, \gamma, \delta, \chi \\
& \quad P_1 \in S_{nz}, P_2 \in S_{ny}, \delta > 0, \chi \geq 0 \\
& \quad \mathcal{M}(0) \leq 0, \mathcal{M}(T_2) \leq 0
\end{align*} \]

- Challenging problem to tackle from a computational standpoint: nonlinear matrix inequalities
The proposed apparatus allows one to include optimization aspects in the design of the observer

- For example

\[
\begin{align*}
\text{minimize} & \quad (-T_2, \gamma^2) \\
\text{s.t.} & \\
& P_1 \in S^{n_z}_+, P_2 \in S^{n_y}_+, \delta > 0, \chi \geq 0 \\
& M(0) \leq 0, M(T_2) \leq 0
\end{align*}
\]

- Challenging problem to tackle from a computational standpoint: nonlinear matrix inequalities

- Whenever \( \delta \) is given, our results allows one to obtain sufficient conditions in the form of linear matrix inequalities
Optimization Aspects

The proposed apparatus allows one to include optimization aspects in the design of the observer

- For example

\[
\begin{align*}
\text{minimize} & \quad P_1, P_2, L, H, \gamma, \delta, \chi (-T_2, \gamma^2) \\
\text{s.t.} & \quad P_1 \in S_{nz}^n, P_2 \in S_{ny}^n, \delta > 0, \chi \geq 0 \\
& \quad M(0) \leq 0, M(T_2) \leq 0
\end{align*}
\]

- Challenging problem to tackle from a computational standpoint: nonlinear matrix inequalities

- Whenever $\delta$ is given, our results allows one to obtain sufficient conditions in the form of linear matrix inequalities

- A suboptimal solution can be obtained via semidefinite programming (SDP) software by performing a line search for the scalar $\delta$. 
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5. Numerical examples
Consider the following model of the flexible one-link manipulator [Spong1987]

\[
\dot{x} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-48.6 & -1.25 & 48.6 & 0 \\
0 & 0 & 0 & 1 \\
19.5 & 0 & -19.5 & 0 \\
\end{bmatrix} x + \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 2 & 0 \\
0 & -3.33 & 0 & 0 \\
-0.33 & 0 & 0 & 0 \\
\end{bmatrix} \sin(x_3) + \begin{bmatrix}
0 \\
2 \\
0 \\
0 \\
\end{bmatrix} w
\]

\[
y = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{bmatrix} x
\]

Picture from [Raff ea., ACC 2008].
The Elastic Manipulator

Consider the following model of the flexible one-link manipulator [Spong1987]

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & -19.5 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -3.33 \end{pmatrix} \sin(x_3) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} w$$

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x$$

$T_2$ can be selected up to 0.3, i.e. an improvement of 200% with respect to [Raff ea., ACC 2008].
Evolution of $\hat{x}$ and $x$ (black).

Evolution of $\theta$ and $y$ (black).
Our conditions can be used to get an estimate of the largest allowable value of $T_2$ (MATI).

- For this example, pick

$$L = \begin{pmatrix}
9.328 & 1 \\
-48.78 & 22.11 \\
-0.0524 & 3.199 \\
19.41 & -0.9032
\end{pmatrix}, \quad H = -CL$$

- Our conditions are feasible for $T_2$ up to $0.1016$

- Conditions in [Postoyan and Nesic TAC 2009] leads to an estimate of the largest allowable value of $T_2$ equal to $1.08 \times 10^{-8}$. 

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Consider the example in [Karafyllis ea. TAC 2009], which is defined by the following data:

\[ A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \end{pmatrix}, L = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \]

- In [Karafyllis et al. TAC 2009] the authors certify convergence for \( T_2 < 0.089 \)
- For the same setup, we certify exp. convergence for \( T_2 \) up to 0.42, i.e., about 5 times larger!
Consider the example in [Karafyllis et al. TAC 2009], which is defined by the following data:

\[
A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 4 \\ 0 \end{pmatrix}
\]

◮ In [Karafyllis et al. TAC 2009] the authors certify convergence for \( T_2 < 0.089 \)

◮ For the same setup, we certify exp. convergence for \( T_2 \) up to 0.42, i.e., about 5 times larger!

◮ our apparatus can be used to determine a tradeoff curve between \( T_2 \) and \( \gamma \)
Tradeoff curves obtained by considering different relaxations: “H-Lin.” (black), “sample-and-hold” (red), “slack variables” (green) and (blue). In these experiments $\lambda_t = 0.01$. 
For $T_2 = 0.41$, by relying on the “H-Linearization” result, one gets:

$$L = \begin{pmatrix} 2.067 \\ -3 \end{pmatrix}, \quad H = -1.384, \quad \gamma = 36$$
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$$L = \begin{pmatrix} 2.067 \\ -3 \end{pmatrix}, H = -1.384, \gamma = 36$$
Conclusion

We provided an observer to exponentially reconstruct the state of a Lipschitz nonlinear system in the presence of sporadic measurements, these results are published in [CDC 2015] in [IEEE TAC 2019]

The analysis is carried out in the hybrid system framework of [Goebel ea. PUP 2012], which provides robustness guarantees

Due to well-posedness of the resulting hybrid systems, state reconstruction is (semiglobally) practically preserved in the presence of small perturbations (implementation sampling, etc...)

The design of the observer is carried out in an LMI framework
Conclusion

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Presented at ECC18

Hybrid Regional Stabilization of Linear Systems with Actuator Saturation and Multi-Rate Samplers

Francesco Ferrante, Ricardo G. Sanfelice, and Sophie Tarbouriech

Abstract—Regional stability analysis of linear systems with multi-rate samplers and actuator saturation is studied. A hybrid controller is used to perform a fusion of measurements sampled at different times. In between sampling events, the controller behaves as a copy of the plant. When a new measurement is available, the controller state undergoes a jump. The resulting system is analyzed in a hybrid system framework. Sufficient conditions in the form of matrix inequalities are given to determine estimates of the basin of attraction of the closed-loop system. Robustness with respect to small parameter uncertainties is discussed. Finally, the effectiveness of the proposed methodology is shown in an example.

area as testified by the large amount of publications in this field [9], [10], [11]. Many approaches have been proposed in the literature to analyze the effect of actuator saturation in (single-rate) sampled-data control systems [12], [13], [14]. However, to the best of our knowledge, no results are found in the literature to deal with actuator saturation in a multi-rate setting.

To fill this gap, in this paper, building on the model-based controller architecture presented in [15], originally proposed for single-rate non-saturated linear systems, we
We are applying our methodology to stability analysis of networked control systems [Heemels et al., TAC 2010]

Application to vehicles platooning under Denial of Service Attacks

[Merco, F. F., Pisu, ACC 19, submitted], [Merco, F. F., Pisu, ECC 19, submitted]
An interesting direction consists of analyzing a blend of two observers:

\[
\begin{align*}
\dot{\hat{z}}(t) &= A\hat{z}(t) + B\psi(S\hat{z}(t)) + L\theta(t) \\
\dot{\theta}(t) &= H\theta(t)
\end{align*}
\] \hspace{1cm} \forall t \neq t_k, k \in \mathbb{N}_{>0}

\[
\begin{align*}
\hat{z}(t^+) &= \hat{z}(t) + R(y(t) - C\hat{z}(t)) \\
\theta(t^+) &= y(t) - C\hat{z}(t)
\end{align*}
\] \hspace{1cm} \forall t = t_k, k \in \mathbb{N}_{>0}

And a lot more...
The End

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